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ON A SUBCLASS OF ANALYTIC AND UNIVALENT FUNCTION DEFINED BY AL-OBOUDI OPERATOR

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ABSTRACT

In the present paper, a subclass of analytic and univalent function is defined by Al-Oboudi Operator and we have obtained among other results like, Coefficient estimates, Growth and distortion theorem, external properties for the classes $T_n S_p^{\lambda}(\alpha,\beta,\xi,\gamma,\delta,A,B)$ and $T_n V^{\lambda}(\alpha,\beta,\xi,\gamma,\delta,A,B)$.

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1. INTRODUCTION

Let S denote the class of functions of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n \tag{1}$$

that are analytic and univalent in the disc $|z| < 1$. For $0 \le \alpha < 1$ $S^*(\alpha)$ and $K(\alpha)$ denote the subfamilies of S consisting of functions starlike of order α and convex of order α respectively.

The subfamily T of S consists of functions of the form

$$f(z) = z - \sum_{n=2}^{\infty} a_n z^n, \qquad a_n \ge 0, \text{for } n = 2,3,\ldots; \quad z \in U. \tag{2}$$

Silverman [6] investigated function in the classes $T^*(\alpha) = T \cap S^*(\alpha)$ and $C(\alpha) = T \cap K(\alpha)$.

Let $n \in N$ and $\lambda \ge 0$. Denote by D_λ^n the Al-Oboudi operator [3] defined by, $D_\lambda^n : A \to A$,

$$D_\lambda^0 f(z) = f(z)$$
$$D_\lambda^1 f(z) = (1-\lambda) f(z) + \lambda z f'(z) = D_\lambda f(z)$$
$$D_\lambda^n f(z) = D_\lambda [D_\lambda^{n-1} f(z)].$$

Note that for f(z) is given by (I),

$$D_\lambda^n f(z) = z + \sum_{j=1}^{\infty} [1+(j-1)\lambda]^\beta a_j z^j \text{ when } \lambda = 1,$$

D_λ^n is the Salagean differential operator $D_\lambda^n : A \to A, n \in N$ defined as:

$$D^0 f(z) = f(z)$$
$$D^1 f(z) = Df(z) = z f^{'}(z)$$
$$D^n f(z) = D[D^{n-1} f(z)].$$

Definition 1.1: [8] Let $\beta, \lambda \in R, \beta \ge 0, \lambda \ge 0$ and $f(z) = z + \sum_{j=2}^{\infty} a_j z^j$ we denote by D_λ^β the linear operator defined by $D_\lambda^\beta : A \to A,\ D_\lambda^\beta\ f(z) = z + \sum_{j=2}^{\infty} [1+(j-1)\lambda]^\beta a_j z^j$.

Remark 1.1: If $f(z) \in T,\quad f(z) = z - \sum_{j=2}^{\infty} a_j z^j,\ a_j \ge 0,\ j = 2,3,...,\ z \in U$

Then $D_\lambda^\beta\ f(z) = z - \sum_{j=2}^{\infty} [1+(j-1)\lambda]^\beta a_j z^j$.

In this paper using the operator D_λ^β we introduce the classes $T_n S_p^\lambda(\alpha, \beta, \xi, \gamma, \delta, A, B)$ and $T_n V^\lambda(\alpha, \beta, \xi, \gamma, \delta, A, B)$ and obtain coefficient estimates for these classes when the functions have negative coefficients. We also obtain growth and distortion theorems, closure theorem for functions in these classes.

Definition 1.2: We say that a function $f(z) \in T$ is in the class $T_n S_p^\lambda(\alpha, \beta, \xi, \gamma, \delta, A, B)$ if and only if

$$\left| \frac{\dfrac{D_\lambda^{\beta+1} f(z)}{D_\lambda^\beta f(z)} - 1}{(B-A)\xi\left(\dfrac{D_\lambda^{\beta+1} f(z)}{D_\lambda^\beta f(z)} - \alpha\right) + A\gamma\left(\dfrac{D_\lambda^{\beta+1} f(z)}{D_\lambda^\beta f(z)} - 1\right)} \right| < \delta$$

Where $|z| < 1, 0 < \delta \le 1, 1/2 \le \xi \le 1, \lambda \ge 0, 0 \le \alpha \le 1/2\xi, 1/2 < \gamma \le 1, \beta \ge 0, 0 < B \le 1, -1 \le A < B \le 1.$

Definition 1.3: A function $f(z) \in T$ is said to belong to the class $T_n S_p^\lambda(\alpha, \beta, \xi, \gamma, \delta, A, B)$ if and only if

$$\left| \frac{\dfrac{D_\lambda^{\beta+2} f(z)}{D_\lambda^{\beta+1} f(z)} - 1}{(B-A)\xi\left(\dfrac{D_\lambda^{\beta+2} f(z)}{D_\lambda^{\beta+1} f(z)} - \alpha\right) + A\gamma\left(\dfrac{D_\lambda^{\beta+2} f(z)}{D_\lambda^{\beta+1} f(z)} - 1\right)} \right| < \delta, \text{ where}$$

$|z| < 1, 0 < \delta \le 1, 1/2 \le \xi \le 1, \lambda \ge 0, 0 \le \alpha \le 1/2\xi, 1/2 < \gamma \le 1, \beta \ge 0, 0 < B \le 1, -1 \le A < B \le 1$

If we replace $\beta = 0, \lambda = 1$ we obtain the corresponding results of S.M. Khairnar and Meena More [4]. If we replace $\beta = 0, \lambda = 1$ and $\gamma = 1$ we obtain the results of Aghalary and Kulkarni [2] and Silverman and Silvia [7]. If we replace $\beta = 0, \lambda = 1$ and $\xi = 1$, we obtain the corresponding results of [9].

2. MAIN RESULTS COEFFICIENT ESTIMATES

Theorem 2.1: A function $f(z) = z - \sum_{j=2}^{\infty} a_j z^j$, $(a_j \geq 0)$ is in $T_n S_p^{\lambda}(\alpha, \beta, \xi, \gamma, \delta, A, B)$ if and only if

$$\sum_{j=2}^{\infty} [1 + (j-1)\lambda]^{\beta} \left[(j-1)\lambda\{1 + A\gamma\delta + (B-A)\delta\xi\} + (B-A)\delta\xi(1-\alpha) \right] a_j \leq (B-A)\delta\xi(1-\alpha)$$

Proof: Suppose,

$$\sum_{j=2}^{\infty} [1 + (j-1)\lambda]^{\beta} \left[(j-1)\lambda\{1 + A\gamma\delta + (B-A)\delta\xi\} + (B-A)\delta\xi(1-\alpha) \right] a_j \leq (B-A)\delta\xi(1-\alpha)$$

we have

$$\left| D_{\lambda}^{\beta+1} f(z) - D_{\lambda}^{\beta} f(z) \right| - \delta \left| (B-A)\xi \left[D_{\lambda}^{\beta+1} f(z) - \alpha D_{\lambda}^{\beta} f(z) \right] + A\gamma \left[D_{\lambda}^{\beta+1} f(z) - D_{\lambda}^{\beta} f(z) \right] \right| < 0$$

with the provision,

$$\left| z - \sum_{j=2}^{\infty} (1+(j-1)\lambda)^{\beta+1} a_j z^j - z + \sum_{j=2}^{\infty} (1+(j-1)\lambda)^{\beta} a_j z^j \right| -$$

$$\delta \left| (B-A)\xi \left[z - \sum_{j=2}^{\infty} (1+(j-1)\lambda)^{\beta+1} a_j z^j - \alpha z + \alpha \sum_{j=2}^{\infty} (1+(j-1)\lambda)^{\beta} a_j z^j \right] \right.$$

$$\left. + A\gamma \left[z - \sum_{j=2}^{\infty} (1+(j-1)\lambda)^{\beta+1} a_j z^j - z + \sum_{j=2}^{\infty} (1+(j-1)\lambda)^{\beta} a_j z^j \right] \right| < 0$$

For $|z| = r < 1$ it is bounded above by

$$\sum_{j=2}^{\infty} [1 + (j-1)\lambda]^{\beta} \left[(j-1)\lambda\{1 + A\gamma + (B-A)\delta\xi\} + (B-A)\delta\xi(1-\alpha) \right] a_j r^j \leq (B-A)\delta\xi(1-\alpha).$$

Hence $f(z) \in T_n S_p^{\lambda}(\alpha, \beta, \xi, \gamma, \delta, A, B)$.

Now we prove the converse result.

Let, $$\left| \frac{\dfrac{D_{\lambda}^{\beta+1} f(z)}{D_{\lambda}^{\beta} f(z)} - 1}{(B-A)\xi \left(\dfrac{D_{\lambda}^{\beta+1} f(z)}{D_{\lambda}^{\beta} f(z)} - \alpha \right) + A\gamma \left(\dfrac{D_{\lambda}^{\beta+1} f(z)}{D_{\lambda}^{\beta} f(z)} - 1 \right)} \right| < \delta$$

$$= \left| \frac{\dfrac{z - \sum_{j=2}^{\infty}(1+(j-1)\lambda)^{\beta+1} a_j z^j}{z - \sum_{j=2}^{\infty}(1+(j-1)\lambda)^{\beta} a_j z^j} - 1}{(B-A)\xi \left[\dfrac{z - \sum_{j=2}^{\infty}(1+(j-1)\lambda)^{\beta+1} a_j z^j}{z - \sum_{j=2}^{\infty}(1+(j-1)\lambda)^{\beta} a_j z^j} - \alpha \right] + A\gamma \left[\dfrac{z - \sum_{j=2}^{\infty}(1+(j-1)\lambda)^{\beta+1} a_j z^j}{z - \sum_{j=2}^{\infty}(1+(j-1)\lambda)^{\beta} a_j z^j} \right] - 1} \right| < \delta$$

$$= \left| \frac{\sum_{j=2}^{\infty}(1+(j-1)\lambda)^{\beta}(j-1)\lambda a_j z^j}{(B-A)\xi z(1-\alpha) - \sum_{j=2}^{\infty}(1+(j-1)\lambda)^{\beta}\left[(B-A)\xi(1-\alpha) + ((B-A)\xi + A\gamma)(j-1)\lambda\right] a_j z^j} \right| < \delta$$

As $|\operatorname{Re} f(z)| \le |z|$ *for all z, we have*

$$\operatorname{Re} \left| \frac{\sum_{j=2}^{\infty}(1+(j-1)\lambda)^{\beta}(-(j-1)\lambda) a_j z^j}{(B-A)\xi z(1-\alpha) - \sum_{j=2}^{\infty}(1+(j-1)\lambda)^{\beta}\left[(B-A)\xi(1-\alpha) + ((B-A)\xi + A\gamma)(j-1)\lambda\right] a_j z^j} \right| < \delta$$

we choose values of z on real axis such that $\dfrac{D_{\lambda}^{\beta+1}}{D_{\lambda}^{\beta}}$ is real and clearing the denominator of above expression and allowing $z \to 1$ through real values, we obtain.

$$\sum_{j=2}^{\infty}[1+(j-1)\lambda]^{\beta}\left[(j-1)\lambda\{1 + A\gamma\delta + (B-A)\delta\xi\} + (B-A)\delta\xi(1-\alpha)\right] a_j \le (B-A)\delta\xi(1-\alpha)$$

$$\Rightarrow \sum_{j=2}^{\infty}\left[(1+(j-1)\lambda)^{\beta}\{(j-1)\lambda(1 + A\gamma\delta + (B-A)\delta\xi) + (B-A)\delta\xi(1-\alpha)\}\right] a_j - (B-A)\delta\xi(1-\alpha) \le 0$$

Remark 2.1: If $f(z) \in T_n S_p^{\lambda}(\alpha,\beta,\xi,\gamma,\delta,A,B)$ then

$$a_j \le \frac{(B-A)\delta\xi(1-\alpha)}{(1+(j-1)\lambda)^{\beta}\{(j-1)\lambda(1 + A\gamma\delta + (B-A)\delta\xi) + (B-A)\delta\xi(1-\alpha)\}}, \; j = 2,3,4,.....$$

and equality holds for,

$$f(z) = z - \frac{(B-A)\delta\xi(1-\alpha)}{(1+(j-1)\lambda)^{\beta}\{(j-1)\lambda(1 + A\gamma\delta + (B-A)\delta\xi) + (B-A)\delta\xi(1-\alpha)\}} z^j.$$

Corollary 2.1: If $f(z) \in T_n S_p^{1}(\alpha,\beta,\xi,\gamma,\delta,-1,1)$ (In particular if $A = -1,\ B = 1$) then we get,

$$a_j \le \frac{2\xi\delta(1-\alpha)}{[1+(j-1)\lambda]^{\beta}\{(j-1)\lambda(2\xi\delta+1-\gamma\delta)+2\delta\xi(1-\alpha)\}},\ j=2,3,4,.....$$

and equality holds for,

$$f(z)=z-\frac{2\xi\delta(1-\alpha)}{[1+(j-1)\lambda]^{\beta}\{(j-1)\lambda(2\xi\delta+1-\gamma\delta)+2\delta\xi(1-\alpha)\}}z^j$$

This corollary is due to [11].

Corollary 2.2: If $f(z)\in T_nS_p^1(\alpha,0,\xi,\gamma,\delta,-1,1)$ (in particular $\beta=0,\ \lambda=1,\ B=1,\ A=-1$) then we get,

$$a_j \le \frac{2\delta\xi(1-\alpha)}{(j-1)-\delta\{2\alpha\xi-2\xi j+\gamma j-\gamma\}},\ j=2,3,4,.....$$

and equality holds for, $f(z)=z-\dfrac{2\xi\delta(1-\alpha)}{(j-1)-\delta\{2\alpha\xi-2\xi j+\gamma j-\gamma\}}z^j$

This corollary is due to [4].

Corollary 2.3: If $f(z)\in T_nS_p^1(\alpha,0,\xi,1,\delta,-1,1)$ (In particular $\beta=0,\lambda=1,\ \gamma=1, A=-1$ and $B=1$) then we get, $a_j \le \dfrac{2\xi\delta(1-\alpha)}{(j-1)-\delta\{2\xi\alpha-2\xi j+j-1\}},\ j=2,3,4,.....$

and equality holds for, $f(z)=z-\dfrac{2\xi\delta(1-\alpha)}{(j-1)-\delta\{2\xi\alpha-2\xi j+j-1\}}z^j.$

This corollary is due to [2] and [7].

Corollary 2.4: If $f(z)\in T_nS_p^1(\alpha,0,1,1,\delta,-1,1)$ (in particular $\beta=0,\lambda=1,\gamma=1,\xi=1,A=-1$ and $B=1$) then we get, $a_j \le \dfrac{2\delta(1-\alpha)}{(j-1)-\delta\{2\alpha-j-1\}},\ j=2,3,4,.....$

and equality holds for, $f(z)=z-\dfrac{2\delta(1-\alpha)}{(j-1)-\delta\{2\alpha-j-1\}}z^j.$

This corollary is due to [9].

Corollary 2.5: If $f(z)\in T_nS_p^1(\alpha,0,1,1,1,-1,1)$ (in Particular $\beta=0,\lambda=1,\gamma=1,\xi=1,\delta=1,A=-1,B=1$) if and only if $\sum_{j=2}^{\infty}(j-\alpha)a_j\le(1-\alpha).$

Theorem 2.2: A function $f(z) = z - \sum_{j=2}^{\infty} a_j z^j, \ (a_j \geq 0)$ is in $T_n S_p^{\lambda}(\alpha, \beta, \xi, \gamma, \delta, A, B)$ if and only if

$$\sum_{j=2}^{\infty}[1+(j-1)\lambda]^{\beta+1}\left[(j-1)\lambda\{1+A\gamma+(B-A)\delta\xi\}+(B-A)\delta\xi(1-\alpha)\right]a_j \leq (B-A)\delta\xi(1-\alpha).$$

Proof: The proof of this theorem is analogous to that of Theorem 1, because a function $f(z) \in T_n V^{\lambda}(\alpha, \beta, \xi, \gamma, \delta, A, B)$ if and only if $zf^{'}(z) \in T_n S_p^{\lambda}(\alpha, \beta, \xi, \gamma, \delta, A, B)$. So it is enough that replacing $\beta \, with \, \beta+1$ in Theorem 2.1.

Remark 2.2: If $f(z) \in T_n V^{\lambda}(\alpha, \beta, \xi, \gamma, \delta, A, B)$ then

$$a_j \leq \frac{(B-A)\delta\xi(1-\alpha)}{[1+(j-1)\lambda]^{\beta+1}\left[(j-1)\lambda\{1+A\gamma+(B-A)\delta\xi\}+(B-A)\delta\xi(1-\alpha)\right]}$$

and equality holds for, $f(z) = z - \dfrac{(B-A)\delta\xi(1-\alpha)}{(1+(j-1)\lambda)^{\beta+1}\{(j-1)\lambda(1+A\gamma\delta+(B-A)\delta\xi)+(B-A)\delta\xi(1-\alpha)\}} z^j.$

Corollary 2.6: If $f(z) \in T_n V^{\lambda}(\alpha, \beta, \xi, \gamma, \delta, -1, 1)$ (in particular $\beta = 0, \lambda = 1, A = -1, B = 1$) then we get,

$$a_j \leq \frac{2\delta\xi(1-\alpha)}{(1+(j-1)\lambda)^{\beta+1}\left[(j-1)\lambda(2\delta\xi+1-\gamma\delta)+2\delta\xi(1-\alpha)\right]}, \ j = 2, 3, 4, ...$$

and equality holds for,

$$f(z) = z - \frac{2\delta\xi(1-\alpha)}{(1+(j-1)\lambda)^{\beta+1}\left[(j-1)\lambda(2\delta\xi+1-\gamma\delta)+2\delta\xi(1-\alpha)\right]} z^j$$

This corollary is due to [11].

Corollary 2.7: If $f(z) \in T_n V^1(\alpha, 0, \xi, \gamma, \delta, -1, 1)$ (in particular $\beta = 0, \lambda = 1, A = -1, B = 1$) then we get,

$$a_j \leq \frac{2\delta\xi(1-\alpha)}{j\left[(j-1)-\delta\{2\alpha\xi-2\xi j+\gamma j-\gamma\}\right]}, \ j = 2, 3, 4, ...$$

and equality holds for, $f(z) = z - \dfrac{2\delta\xi(1-\alpha)}{j\left[(j-1)-\delta\{2\alpha\xi-2\xi j+\gamma j-\gamma\}\right]} z^j.$

This corollary is due to [4].

Corollary 2.8: If $f(z) \in T_n V^1(\alpha, 0, \xi, 1, \delta, -1, 1)$ (in particular $\beta = 0, \lambda = 1, \ \gamma = 1, \ A = -1$ and $B = 1$) then we get, $a_j \leq \dfrac{2\delta\xi(1-\alpha)}{j\left[(j-1)-\delta\{2\alpha\xi-2\xi j+j-1\}\right]}, \ j = 2, 3, 4, ...$

and equality holds for, $f(z)=z-\frac{2\delta\xi(1-\alpha)}{j\left[(j-1)-\delta\{2\alpha\xi-2\xi j+j-1\}\right]}z^{j}.$

This corollary is due to [2] and [7].

Corollary 2.9: If $f(z)\in T_nV^1(\alpha,0,1,1,\delta,-1,1)$ (in particular $\beta=0,\lambda=1,\gamma=1,\ \xi=1,\ A=-1$ and $B=1$)

then we get, $a_j\leq\frac{2\delta(1-\alpha)}{j\left[(j-1)-\delta\{2\alpha-j-1\}\right]}$, $j=2,3,4,.....$

and equality holds for, $f(z)=z-\frac{2\delta(1-\alpha)}{j\left[(j-1)-\delta\{2\alpha-j-1\}\right]}\ z^{j}.$

Corollary 2.10: If $f(z)\in T_nv^1(\alpha,0,1,1,1,-1,1)$ (in particular $\beta=0,\lambda=1,\gamma=1,\xi=1,\ \delta=1,A=-1$ and $B=1$)

if and only if $\sum_{j=2}^{\infty}j\left(j-\alpha\right)a_j\leq\left(1-\alpha\right).$

3. GROWTH AND DISTORTION THEOREM

Theorem 3.1: If $f(z)\in T_nS_p^{\lambda}(\alpha,\beta,\xi,\gamma,\delta,A,B)$ then

$$r-r^2\left[\frac{(B-A)\xi\delta(1-\alpha)}{(1+\lambda)^{\beta}\{\lambda+(B-A)\xi\delta\lambda+A\gamma\delta\lambda+(B-A)\xi\delta(1-\alpha)\}}\right]\leq\left|f(z)\right|$$
$$\leq r+r^2\left[\frac{(B-A)\xi\delta(1-\alpha)}{(1+\lambda)^{\beta}\{\lambda+(B-A)\xi\delta\lambda+A\gamma\delta\lambda+(B-A)\xi\delta(1-\alpha)\}}\right]$$

Equality holds for $f(z)=z-\frac{(B-A)\delta\xi(1-\alpha)}{\{1+2(B-A)\xi\delta\}+\delta\{A\gamma-(B-A)\xi\alpha\}}z^2\ \ at\ \ z\pm r$

Proof: By theorem 2.1, we have $f(z)\in T_nS_p^{\lambda}(\alpha,\beta,\xi,\gamma,\delta,A,B)$ if and only if

$$\sum_{j=2}^{\infty}[1+(j-1)\lambda]^{\beta}\left[(j-1)\lambda\{1+A\gamma\delta+(B-A)\delta\xi\}+(B-A)\delta\xi(1-\alpha)\right]a_j\leq(B-A)\delta\xi(1-\alpha)$$

Let, $t=1-\frac{(B-A)\xi\delta(1-\alpha)}{\lambda+(B-A)\xi\delta\lambda+A\gamma\delta\lambda}$

$$\therefore\ f(z)\in T_nS_p^{\lambda}(\alpha,\beta,\xi,\gamma,\delta,A,B)\text{ if and only if }\sum_{j=2}^{\infty}(1+(j-1)\lambda)^{\beta}(j-t)a_j\leq(1-t)\qquad(3)$$

when j=2

$$(1+\lambda)^{\beta}(2-t)\sum_{j=2}^{\infty} a_j \le \sum_{j=2}^{\infty}(1+(j-1)^{\beta} a_j (j-t) \le (1-t)$$

$$\therefore |f(z)| \le r + \sum_{j=2}^{\infty} a_n r^n \le r + r^2 \sum_{j=2}^{\infty} a_n \le r + r^2 \left[\frac{1-t}{(1+\lambda)^{\beta}(2-t)}\right]$$

similarly,

$$\therefore |f(z)| \ge r - \sum_{j=2}^{\infty} a_n r^n \ge r - r^2 \sum_{j=2}^{\infty} a_n \ge r - r^2 \left[\frac{1-t}{(1+\lambda)^{\beta}(2-t)}\right]$$

so,

$$r - r^2\left[\frac{1-t}{(1+\lambda)^{\beta}(2-t)}\right] \le |f(z)| \le r + r^2\left[\frac{1-t}{(1+\lambda)^{\beta}(2-t)}\right].$$

Hence the result.

i.e.

$$r - r^2\left[\frac{(B-A)\delta\xi(1-\alpha)}{(1+\lambda)^{\beta}\{\lambda+(B-A)\xi\delta\lambda+A\gamma\delta\lambda+(B-A)\xi\delta(1-\alpha)\}}\right] \le |f(z)|$$

$$\le r + r^2\left[\frac{(B-A)\delta\xi(1-\alpha)}{(1+\lambda)^{\beta}\{\lambda+(B-A)\xi\delta\lambda+A\gamma\delta\lambda+(B-A)\xi\delta(1-\alpha)\}}\right]$$

Corollary 3.1: If $f(z) \in T_n S_p^1(\alpha, 0, \xi, \gamma, \delta, A, B)$ (i.e. replacing $\beta = 0, \lambda = 1$) then we get,

$$r - r^2\left[\frac{(B-A)\delta\xi(1-\alpha)}{1+2(B-A)\xi\delta+\delta\{A\gamma-(B-A)\xi\alpha\}}\right] \le |f(z)| \le r + r^2\left[\frac{(B-A)\delta\xi(1-\alpha)}{1+2(B-A)\xi\delta+\delta\{A\gamma-(B-A)\xi\alpha\}}\right]$$

and equality holds for $f(z) = z - \dfrac{(B-A)\delta\xi(1-\alpha)}{1+2(B-A)\xi\delta+\delta\{A\gamma-(B-A)\xi\alpha\}} z^2 \quad at \quad z = \pm r$

This corollary is due to [4].

Corollary 3.2: If $f(z) \in T_n S_p^1(\alpha, 0, \xi, 1, \delta, A, B)$ (i.e. replacing $\beta = 0, \lambda = 1 and\ \gamma = 1$) then we get,

$$r - r^2\left[\frac{(B-A)\delta\xi(1-\alpha)}{1+2(B-A)\xi\delta+\delta\{A-(B-A)\xi\alpha\}}\right] \le |f(z)| \le r + r^2\left[\frac{(B-A)\delta\xi(1-\alpha)}{1+2(B-A)\xi\delta+\delta\{A-(B-A)\xi\alpha\}}\right]$$

and Equality holds for, $f(z) = z - \dfrac{(B-A)\delta\xi(1-\alpha)}{1+2(B-A)\xi\delta+\delta\{A-(B-A)\xi\alpha\}} z^j \quad at \quad z = \pm r$

This corollary is due to [2] and [7].

Corollary 3.3: If $f(z) \in T_n S_p^1(\alpha, 0, 1, 1, \delta, A, B)$ (i.e. replacing $\beta = 0, \lambda = 1, \gamma = 1 and\ \xi = 1$) then we get,

$$r-r^{2}\left[\frac{(B-A)\delta(1-\alpha)}{1+2(B-A)\delta+\delta\{A-(B-A)\alpha\}}\right]\le|f(z)|\le r+r^{2}\left[\frac{(B-A)\delta(1-\alpha)}{1+2(B-A)\delta+\delta\{A-(B-A)\alpha\}}\right]$$

and Equality holds for,

$$f(z)=z-\frac{(B-A)\delta(1-\alpha)}{1+2(B-A)\delta+\delta\{A-(B-A)\alpha\}}z^{j}\quad at\quad z=\pm r$$

This corollary is due to [9].

Theorem 3.2: If $f(z)\in T_nV^{\lambda}(\alpha,\beta,\xi,\gamma,\delta,A,B)$ then

$$r-r^{2}\left[\frac{(B-A)\xi\delta(1-\alpha)}{(1+\lambda)^{\beta+1}\{\lambda+(B-A)\xi\delta\lambda+A\gamma\delta\lambda+(B-A)\xi\delta(1-\alpha)\}}\right]\le|f(z)|$$

$$\le r+r^{2}\left[\frac{(B-A)\xi\delta(1-\alpha)}{(1+\lambda)^{\beta+1}\{\lambda+(B-A)\xi\delta\lambda+A\gamma\delta\lambda+(B-A)\xi\delta(1-\alpha)\}}\right]$$

Proof: The proof of this theorem is analogous to that of theorem 3.1, because a function $f(z)\in T_nV^{\lambda}(\alpha,\beta,\xi,\gamma,\delta,A,B)$ if and only if $zf'(z)\in T_nS_p^{\lambda}(\alpha,\beta,\xi,\gamma,\delta,A,B)$. So it is enough that replacing $\beta\ with\ \beta+1$ in Theorem: 2.1.

Corollary 3.4: If $f(z)\in T_nV^{1}(\alpha,0,\xi,\gamma,\delta,A,B)$ (i.e. replacing $\beta=0,\lambda=1$) then we get,

$$r-r^{2}\left[\frac{(B-A)\delta\xi(1-\alpha)}{2[1+2(B-A)\xi\delta+\delta\{A\gamma-(B-A)\xi\alpha)\}]}\right]\le|f(z)|\le r+r^{2}\left[\frac{(B-A)\delta\xi(1-\alpha)}{2[1+2(B-A)\xi\delta+\delta\{A\gamma-(B-A)\xi\alpha)\}]}\right]$$

And equality holds for

$$f(z)=z-\left[\frac{(B-A)\delta\xi(1-\alpha)}{2[1+2(B-A)\xi\delta+\delta\{A\gamma-(B-A)\xi\alpha)\}]}\right]z^{j}\quad at\quad z=\pm r$$

This corollary is due to [4].

Corollary 3.5: If $f(z)\in T_nV^{1}(\alpha,0,\xi,1,\delta,A,B)$ (i.e. replacing $\beta=0,\lambda=1\,and\ \gamma=1$) then we get,

$$r-r^{2}\left[\frac{(B-A)\delta\xi(1-\alpha)}{2[1+2(B-A)\xi\delta+\delta\{A-(B-A)\xi\alpha)\}]}\right]\le|f(z)|\le r+r^{2}\left[\frac{(B-A)\delta\xi(1-\alpha)}{2[1+2(B-A)\xi\delta+\delta\{A-(B-A)\xi\alpha)\}]}\right]$$

And Equality holds for,

$$f(z)=z-\left[\frac{(B-A)\delta\xi(1-\alpha)}{2[1+2(B-A)\xi\delta+\delta\{A-(B-A)\xi\alpha)\}]}\right]z^{j}\quad at\quad z=\pm r$$

This corollary is due to [2] and [7].

Corollary 3.6: If $f(z)\in T_nV^1(\alpha,0,1,1,\delta,A,B)$ (i.e. replacing $\beta=0,\lambda=1,\gamma=1 and \xi=1$) then we get,

$$r-r^2\left[\frac{(B-A)\delta(1-\alpha)}{2[1+2(B-A)\delta+\delta\{A-(B-A)\alpha)\}]}\right]\le|f(z)|\le r+r^2\left[\frac{(B-A)\delta(1-\alpha)}{2[1+2(B-A)\delta+\delta\{A-(B-A)\alpha)\}]}\right]$$

and Equality holds for,

$$f(z)=z-\frac{(B-A)\delta(1-\alpha)}{2[1+2(B-A)\delta+\delta\{A-(B-A)\alpha)\}]}z^j \quad at \quad z=\pm r$$

This corollary is due to [9].

Theorem 3.3: If $f(z)\in T_nS_p^\lambda(\alpha,\beta,\xi,\gamma,\delta,A,B)$ then

$$1-r\left[\frac{(B-A)^2\xi\delta(1-\alpha)}{(1+\lambda)^\beta\{\lambda(1+A\gamma\delta)+(B-A)\xi\delta(\lambda+1-\alpha)\}}\right]\le|f(z)|$$

$$\le 1+r^2\left[\frac{(B-A)^2\xi\delta(1-\alpha)}{(1+\lambda)^\beta\{\lambda(1+A\gamma\delta)+(B-A)\xi\delta(\lambda+1-\alpha)\}}\right]$$

Proof: Since $f(z)\in T_nS_p^\lambda(\alpha,\beta,\xi,\gamma,\delta,A,B)$ we have by Theorem.3.1,

$$\sum_{j=2}^{\infty}(1+(j-1)\lambda)^\beta(j-t)a_j\le(1-t)$$

In view of Theorem 3.1 we have

$$\sum_{j=2}^{\infty} ja_j=\sum_{j=2}^{\infty}(j-1)a_j+t\sum_{j=2}^{\infty}a_j\le\frac{(B-A)(1-t)}{(1+\lambda)^\beta(2-t)} \tag{4}$$

$$|f'(z)|\le 1+\sum_{n=2}^{\infty}na_n|z|^{n-1}\le 1+r\sum_{n=2}^{\infty}na_n\le 1+r\left[\frac{(B-A)(1-t)}{(1+\lambda)^\beta(2-t)}\right]$$

Similarly,

$$|f'(z)|\ge 1-\sum_{n=2}^{\infty}na_n|z|^{n-1}\ge 1-r\sum_{n=2}^{\infty}na_n\ge 1-r\left[\frac{(B-A)(1-t)}{(1+\lambda)^\beta(2-t)}\right]$$

So, $1-r\left[\frac{(B-A)(1-t)}{(1+\lambda)^\beta(2-t)}\right]\le|f'(z)|\le 1+r\left[\frac{(B-A)(1-t)}{(1+\lambda)^\beta(2-t)}\right]$

Substituting the value of t in above inequality,

$$1-r\left[\frac{(B-A)^2\xi\delta(1-\alpha)}{(1+\lambda)^{\beta}\{\lambda(1+A\gamma\delta)+(B-A)\xi\delta(\lambda+1-\alpha)\}}\right]\leq\left|f^{'}(z)\right|$$
$$\leq 1+r\left[\frac{(B-A)^2\xi\delta(1-\alpha)}{(1+\lambda)^{\beta}\{\lambda(1+A\gamma\delta)+(B-A)\xi\delta(\lambda+1-\alpha)\}}\right]$$

Corollary 3.7: If $f(z)\in T_nS_p^1(\alpha,0,\xi,\gamma,\delta,A,B)$ (i.e. replacing $\beta=0,\lambda=1$) then we get,

$$1-r\left[\frac{(B-A)^2\xi\delta(1-\alpha)}{(1+A\gamma\delta)+(B-A)\xi\delta(2-\alpha)\}}\right]\leq\left|f^{'}(z)\right|\leq 1+r\left[\frac{(B-A)^2\xi\delta(1-\alpha)}{(1+A\gamma\delta)+(B-A)\xi\delta(2-\alpha)\}}\right] \text{ for } |z|=r$$

This corollary is due to [4].

Corollary 3.8: If $f(z)\in T_nS_p^1(\alpha,0,\xi,1,\delta,A,B)$ (i.e. replacing $\beta=0,\lambda=1 and\ \gamma=1$) then we get,

$$1-r\left[\frac{(B-A)^2\xi\delta(1-\alpha)}{(1+A\delta)+(B-A)\xi\delta(2-\alpha)\}}\right]\leq\left|f^{'}(z)\right|\leq 1+r\left[\frac{(B-A)^2\xi\delta(1-\alpha)}{(1+A\delta)+(B-A)\xi\delta(2-\alpha)\}}\right] \text{ for } |z|=r$$

This corollary is due to [2] and [7].

Corollary 3.9: If $f(z)\in T_nS_p^1(\alpha,0,1,1,\delta,A,B)$ (i.e. replacing $\beta=0,\lambda=1,\gamma=1 and\ \xi=1$) then we get,

$$1-r\left[\frac{(B-A)^2\delta(1-\alpha)}{(1+A\delta)+(B-A)\delta(2-\alpha)\}}\right]\leq\left|f^{'}(z)\right|\leq 1+r\left[\frac{(B-A)^2\delta(1-\alpha)}{(1+A\delta)+(B-A)\delta(2-\alpha)\}}\right] \text{ for } |z|=r$$

This corollary is due to [9].

Theorem 3.4: If $f(z)\in T_nV^{\lambda}(\alpha,\beta,\xi,\gamma,\delta,A,B)$ then

$$1-r\left[\frac{(B-A)^2\xi\delta(1-\alpha)}{(1+\lambda)^{\beta+1}\{\lambda(1+A\gamma\delta)+(B-A)\xi\delta(\lambda+1-\alpha)\}}\right]\leq\left|f(z)\right|$$
$$\leq 1+r^2\left[\frac{(B-A)^2\xi\delta(1-\alpha)}{(1+\lambda)^{\beta+1}\{\lambda(1+A\gamma\delta)+(B-A)\xi\delta(\lambda+1-\alpha)\}}\right]$$

Proof: The proof of this theorem is analogous to that of theorem 3.3, because a function $f(z)\in T_nV^{\lambda}(\alpha,\beta,\xi,\gamma,\delta,A,B)$ if and only if $zf^{'}(z)\in T_nS_p^{\lambda}(\alpha,\beta,\xi,\gamma,\delta,A,B)$. So it is enough that replacing $\beta\ with\ \beta+1$ in Theorem: 3.3.

Corollary 3.10: If $f(z)\in T_nV^1(\alpha,0,\xi,\gamma,\delta,A,B)$ (i.e. replacing $\beta=0,\lambda=1$) then we get,

$$1-r\left[\frac{(B-A)^2\xi\delta(1-\alpha)}{(1+A\gamma\delta)+(B-A)\xi\delta(2-\alpha)\}}\right]\leq\left|f(z)\right|\leq 1+r^2\left[\frac{(B-A)^2\xi\delta(1-\alpha)}{(1+A\gamma\delta)+(B-A)\xi\delta(2-\alpha)\}}\right]\quad for\,|z|=r$$

This corollary is due to [4].

Corollary 3.11: If $f(z) \in T_n V^1(\alpha, 0, \xi, 1, \delta, A, B)$ (i.e. replacing $\beta = 0, \lambda = 1 \, and \; \gamma = 1$) then we get,

$$1 - r\left[\frac{(B-A)^2 \xi\delta(1-\alpha)}{(1+A\delta)+(B-A)\xi\delta(2-\alpha)\}}\right] \le |f(z)| \le 1 + r^2\left[\frac{(B-A)^2 \xi\delta(1-\alpha)}{(1+A\delta)+(B-A)\xi\delta(2-\alpha)\}}\right] \quad for\,|z| = r$$

This corollary is due to [2] and [7].

Corollary 3.12: If $f(z) \in T_n V^1(\alpha, 0, 1, 1, \delta, A, B)$ (i.e. replacing $\beta = 0, \lambda = 1, \gamma = 1 \, and \; \xi = 1$) then we get,

$$1 - r\left[\frac{(B-A)^2 \delta(1-\alpha)}{(1+A\delta)+(B-A)\delta(2-\alpha)\}}\right] \le |f(z)| \le 1 + r^2\left[\frac{(B-A)^2 \delta(1-\alpha)}{(1+A\delta)+(B-A)\delta(2-\alpha)\}}\right] \quad for\,|z| = r$$

This corollary is due to [9].

4. CLOSURE THEOREM

Theorem 4.1: Let $f_1(z) = z$ and

$$f_j(z) = \frac{(B-A)\delta\xi(1-\alpha)}{[1+(j-1)\lambda]^\beta[(j-1)\lambda\{1+A\gamma\delta+(B-A)\delta\xi\}+(B-A)\delta\xi(1-\alpha)]} z^j \quad for\; j = 2, 3, 4, ...$$

Then $f(z) \in T_n V^\lambda(\alpha, \beta, \xi, \gamma, \delta, A, B)$ if and only if f(z) can be expressed in the forms

$$f(z) = \sum_{j=1}^{\infty} \lambda_j f_j(z), \; where \; \lambda_j \ge 0 \;\; and \;\; \sum_{j=1}^{\infty} \lambda_j = 1$$

Proof: Let $f(z) = \sum_{j=1}^{\infty} \lambda_j f_j(z), \lambda_j \ge 0\,,\; j = 1, 2, 3, ...\; with \sum_{j=1}^{\infty} \lambda_j = 1$

We have, $f(z) = \sum_{j=1}^{\infty} \lambda_j f_j(z) = \lambda_1 f_1(z) + \sum_{j=2}^{\infty} \lambda_j f_j(z)$

$$\therefore f(z) = z - \sum_{j=2}^{\infty} \lambda_j \left[\frac{(B-A)\delta\xi(1-\alpha)}{[1+(j-1)\lambda]^\beta[(j-1)\lambda\{1+A\gamma\delta+(B-A)\delta\xi\}+(B-A)\delta\xi(1-\alpha)]} z^j\right]$$

Then,

$$\sum_{j=2}^{\infty} \lambda_j \left[\frac{(B-A)\delta\xi(1-\alpha)}{[1+(j-1)\lambda]^\beta[(j-1)\lambda\{1+A\gamma\delta+(B-A)\delta\xi\}+(B-A)\delta\xi(1-\alpha)]} \times \frac{[1+(j-1)\lambda]^\beta[(j-1)\lambda\{1+A\gamma\delta+(B-A)\delta\xi\}+(B-A)\delta\xi(1-\alpha)]}{(B-A)\delta\xi(1-\alpha)}\right] = \sum_{j=2}^{\infty} \lambda_j = 1 - \lambda \le 1$$

Hence, $f(z) \in T_n V^\lambda(\alpha, \beta, \xi, \gamma, \delta, A, B)$

Conversely, suppose $f(z) \in T_n V^{\lambda}(\alpha, \beta, \xi, \gamma, \delta, A, B)$ then remark of theorem 2.1 gives us

$$a_j \le \frac{(B-A)\delta\xi(1-\alpha)}{(1+(j-1)\lambda)^{\beta}\{(j-1)\lambda(1+A\gamma\delta+(B-A)\delta\xi)+(B-A)\delta\xi(1-\alpha)\}}, \; j=2,3,4,.....$$

We take $\lambda_j = \frac{(1+(j-1)\lambda)^{\beta}\{(j-1)\lambda(1+A\gamma\delta+(B-A)\delta\xi)+(B-A)\delta\xi(1-\alpha)\}}{(B-A)\delta\xi(1-\alpha)} a_j, \quad j=2,3,4,.....$

And $\lambda_1 = 1-\sum_{j=1}^{\infty}\lambda_j.$

Then, $f(z) = \sum_{j=1}^{\infty}\lambda_j f_j(z).$

Corollary 4.1: If $f_1(z) = z$ and

$$f_j(z) = z - \frac{(B-A)\delta\xi(1-\alpha)}{(j-1)-\delta\{(B-A)\xi\alpha-(B-A)\xi j - A\gamma(j-1)\}} z^j \;\; for \; j=2,3,4,.....$$

Then $f(z) \in T_n V^1(\alpha, 0, \xi, \gamma, \delta, A, B)$ if and only if f(z) can be expressed in the form

$$f(z) = \sum_{j=1}^{\infty}\lambda_j f_j(z), \; where \; \lambda_j \ge 0 \;\; and \;\; \sum_{j=1}^{\infty}\lambda_j = 1, \; for \; j=1,2,3,4,.....$$

This corollary is due to [4].

Corollary 4.2: If $f_1(z) = z$ and

$$f_j(z) = z - \frac{(B-A)\delta\xi(1-\alpha)}{(j-1)-\delta\{(B-A)\xi\alpha-(B-A)\xi j - A(j-1)\}} z^j \;\; for \; j=2,3,4,.....$$

Then $f(z) \in T_n V^1(\alpha, 0, \xi, 1, \delta, A, B)$ if and only if f(z) can be expressed in the form

$$f(z) = \sum_{j=1}^{\infty}\lambda_j f_j(z), \; where \; \sum_{j=1}^{\infty}\lambda_j = 1 \; and \; \lambda_j \ge 0 \; , \; for \; j=1,2,3,4,.....$$

This corollary is due to [2] and [7].

Corollary 4.3: If $f_1(z) = z$ and $f_j(z) = z - \frac{(B-A)\delta(1-\alpha)}{(j-1)-\delta\{(B-A)\alpha - Bj + A\}} z^j \;\; for \; j=2,3,4,.....$

Then $f(z) \in T_n V^1(\alpha, 0, 1, 1, \delta, A, B)$ if and only if f(z) can be expressed in the form

$$f(z) = \sum_{j=1}^{\infty}\lambda_j f_j(z), \; where \; \sum_{j=1}^{\infty}\lambda_j = 1 \; and \; \lambda_j \ge 0, \; for \; j=1,2,3,4,.....$$

This corollary is due to [9].

Corollary 4.4: If $f_1(z)=z$ and $f_j(z)=z-\frac{(B-A)}{j(B+1)-(A+1)}z^j \quad for\ j=2,3,4,.....$

Then $f(z)\in T_nV^1(0,0,1,1,1,A,B)$ if and only if f(z) can be expressed in the form

$$f(z)=\sum_{j=1}^{\infty}\lambda_j f_j(z),\ \ where\ \sum_{j=1}^{\infty}\lambda_j=1\ and\ \lambda_j\ge 0,\ \ for\ j=1,2,3,4,.....$$

5. CONCLUSIONS

In this paper making use of Al-Oboudi operator two new sub classes of analytic and univalent functions are introduced for the functions with negative coefficients. Many subclasses which are already studied by various researchers are obtained as special cases of our two new sub classes. We have obtained varies properties such as coefficient estimates, growth distortion theorems, Further new subclasses may be possible from the two classes introduced in this paper.

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